

MONGE-AMÈRE MEASURES ON SUBVARIETIES

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ABSTRACT. In this article we address the question whether the complex Monge-Ampère equation is solvable for measures with large singular part. We prove that under some conditions there are no solution when the right-hand side is carried by a smooth subvariety in \mathbb{C}^n of dimension $k < n$.

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1. INTRODUCTION

In this article we study the complex Monge-Ampère equation

$$(dd^c u)^n = \mu \quad (1.1)$$

where μ is a given non-negative Radon measure and $(dd^c \cdot)^n$ denotes the complex Monge-Ampère operator. Monge-Ampère techniques have an interesting history with applications ranging from algebraic and complex geometry to dynamics and theoretical physics (see e.g. [2, 6, 15, 16, 17, 18]). For an historical account of the complex Monge-Ampère operator we refer to [20, 25].

In the seminal article [4], by Bedford and Taylor it was proved that if u is a continuous plurisubharmonic function defined on $\Omega \subset \mathbb{C}^n$, then the left-hand side $(dd^c u)^n$ of the Monge-Ampère equation can not charge on any subvariety in Ω of dimension $k < n$. On the other hand, they show that $(dd^c u)^n$ can charge at a single point and that (1.1) have (in this case) no unique solution. Several author have studied the case when μ is given by a single point mass or a finite sum of such (see e.g. [8, 12, 21, 26, 27]). In [1], a measure μ was constructed that do not have any atoms and it is supported by a pluripolar set such that the equation (1.1) have a solution with this given measure. Hence, there exists a measure μ with large singular part for which equation (1.1) is solvable. The case when the measure μ vanishes on all pluripolar subsets of Ω was completed in [10] (see also [1]).

The growing use of complex Monge-Ampère techniques in applications imply a growing demand on knowledge of (1.1) with a large singular part of the given right-hand side (see e.g. [30, 31]). Therefore, we address in this article the following question:

Aim: *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and let S be smooth subvariety in Ω of dimension $k < n$. Assume that μ is a non-negative Radon measure (not identically zero) defined on S with finite total mass. Do there exists a plurisubharmonic function such that $(dd^c u)^n = \mu$? (with suitable interpretation of dimensions)*

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Now let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain, and let $\mathcal{E}(X)$ be the largest subset of non-positive plurisubharmonic functions u defined on a complex manifold X (e.g. $X = \Omega$) for which $(dd^c u)^n$ is a well-defined non-negative Radon measure. Furthermore, let $\mathcal{F}(\Omega) \subset \mathcal{E}(\Omega)$ be the subset with finite total mass and essential boundary values zero (see section 2 for details). Our question in hand is of a purely local nature, and therefore we can without loss of generality assume that $S = \Delta^k \times \{0\}^{n-k}$, where $\Delta \subset \mathbb{C}$ is the unit disc. Furthermore, for our purpose it is sufficient to make the assumption that the total mass of μ in (1.1) is finite.

From Theorem 3.3 it follows that if there exists a function $\varphi \in \mathcal{E}(S)$ such that $(dd^c \varphi)^k(\{\varphi > -\infty\}) = 0$, then there exists a function $u \in \mathcal{F}(\Omega)$ such that

$$(dd^c u)^n = \mu \times \delta_0^{n-k}$$

with $\mu = (dd^c \varphi)^k$. Here δ_0 denotes the dirac measure at the origin of Δ . It should be emphasized that if $u \in \mathcal{E}(\Omega)$ and $u|_S$ is not identically $-\infty$, then by Theorem 5.11 in [10], we have that $(dd^c u)^n(\{S \setminus \{u = -\infty\}\}) = 0$, and therefore there exists a pluripolar Borel set E in S such that $(dd^c u)^n(S \setminus E) = 0$. Example 2.1 shows that this question is more involved. We construct a function $u \in \mathcal{F}(\Omega)$ such that $(dd^c u)^n = \delta_0$, and $\overline{\{u = -\infty\}} = \Omega$. To show that the situation is even more intricate we construct in Example 4.6 an example of a non-positive plurisubharmonic function u with $u(z) > -\infty$ for all z , but $(dd^c u)^n$ is not a well-defined Radon measure.

We end this article in section 5 by proving the following: *Assume that μ is a non-negative Radon measure defined on Δ^k with finite total mass such that it vanishes on all pluripolar sets in Δ^k . Then there exists no function $u \in \mathcal{E}(\Delta^n)$ such that $u(z', z'') = u(z', |z_{k+1}|, \dots, |z_n|)$ and $(dd^c u)^n = \mu \times \delta_0^{n-k}$. Here we have that $z' = (z_1, \dots, z_k)$.*

For further information on pluripotential theory we refer to [22, 23, 24]

2. PRELIMINARIES

Following the notation introduced by the second-named author in [9, 10] for a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$ we define:

$$\begin{aligned} \mathcal{E}_0(\Omega) &= \left\{ \varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega) : \lim_{z \rightarrow \partial\Omega} \varphi(z) = 0, \int_{\Omega} (dd^c \varphi)^n < \infty \right\}, \\ \mathcal{F}(\Omega) &= \left\{ \varphi \in \mathcal{PSH}(\Omega) : \exists \{u_j\} \subset \mathcal{E}_0(\Omega), \varphi_j \searrow \varphi, \sup_j \int_{\Omega} (dd^c \varphi_j)^n < \infty \right\}, \\ \mathcal{E}(\Omega) &= \left\{ \varphi \in \mathcal{PSH}(\Omega) : \forall \omega \Subset \Omega \exists \varphi_\omega \in \mathcal{F}(\Omega) \text{ such that } \varphi_\omega = \varphi \text{ on } \omega \right\}. \end{aligned}$$

We also need the following generalization to a complex manifold X :

$$\mathcal{E}(X) = \left\{ u \in \mathcal{PSH}(X) : z \in X \text{ there exist a neighbourhood } W \text{ of } z \right. \\ \left. \text{such that } u \in \mathcal{E}(W) \right\}.$$

In the following example we show that there exists a function $u \in \mathcal{F}(\Omega)$ such that $(dd^c u)^n = \delta_0$, and $\overline{\{u = -\infty\}} = \Omega$.

Example 2.1. Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain in \mathbb{C}^n . This example shows that there exists a function $u \in \mathcal{F}(\Omega)$ such that $(dd^c u)^n = \delta_0$, and $\overline{\{u = -\infty\}} = \Omega$.

Step 1: For $j \geq 1$, $1 \leq m \leq n$ let $\{a_{mj}\}_{j \geq 1}$, $a_{mj} > 0$, be sequences of real numbers such that

$$\sum_{j=1}^{\infty} (a_{1j} \cdots a_{nj})^{\frac{1}{n}} < +\infty,$$

and

$$\sum_{j=1}^{\infty} \min(a_{1j}, \dots, a_{m-1j}, a_{m+1j}, \dots, a_{nj}) = \infty \quad \text{for all } 1 \leq m \leq n.$$

To simplify the notation let $A(j) = \min(a_{1j}, \dots, a_{m-1j}, a_{m+1j}, \dots, a_{nj})$. Set

$$u(z) = \sum_{j=1}^{\infty} \max(a_{1j} \ln |z_1|, \dots, a_{nj} \ln |z_n|).$$

Then we have that $u \in \mathcal{F}(\Delta^n)$, and $(dd^c u)^n = c\delta_0$ for some

$$c \in \left[\sum_{j=1}^{\infty} a_{1j} \cdots a_{nj}, \left(\left(\sum_{j=1}^{\infty} (a_{1j} \cdots a_{nj})^{\frac{1}{n}} \right)^n \right) \right].$$

Furthermore, we have that

$$\begin{aligned} & u(z_1, \dots, z_{m-1}, 0, z_{m+1}, \dots, z_n) \\ & \leq \sum_{j=1}^{\infty} A(j) \max(\log |z_1|, \dots, \log |z_{m-1}|, \log |z_{m+1}|, \dots, \log |z_n|) = -\infty. \end{aligned}$$

Hence,

$$\{u = -\infty\} = \{0\} \times \Delta^{n-1} \cup \dots \cup \Delta^{n-1} \times \{0\}.$$

Step 2: We can assume that the unit ball \mathbb{B} is contained in Ω . Let $\{S_j\}$ be a family of hyperplanes such that $\overline{\bigcup_{j=1}^{\infty} (S_j \cap \mathbb{B})} = \overline{\mathbb{B}}$. By using *step 1* together with changing coordinates we can choose $\varphi_j \in \mathcal{F}(\mathbb{B})$ such that

$$(dd^c \varphi_j)^n = \frac{1}{2^j} \delta_0 \quad \text{and} \quad \varphi_j|_{S_j \cap \mathbb{B}} = -\infty.$$

Set

$$\psi = \sum_{j=1}^{\infty} \varphi_j.$$

Then $\psi \in \mathcal{F}(\mathbb{B})$, $\psi|_{S_j \cap \mathbb{B}} = -\infty$ for all j , and $(dd^c \psi)^n \geq \delta_0$. Set

$$\psi^r = \sup\{\Phi \in \mathcal{PSH}(\mathbb{B}) : \Phi \leq 0 \text{ and } \Phi \leq \psi \text{ on } \mathbb{B}(0, r)\}.$$

Here $\mathbb{B}(0, r) \subset \mathbb{C}^n$ is the ball with radius r . This construction yields that $\{\psi^r\}$ increases pointwise to a function $\varphi \in \mathcal{F}(\mathbb{B})$ and $(dd^c \varphi)^n = c\delta_0$, $c > 0$. From the fact that $\psi^r \leq \varphi_j$ on $\mathbb{B}(0, r)$ and $(dd^c \varphi)^n = 0$ on $\mathbb{B} \setminus \{0\}$, we get that $\varphi \leq \varphi_j$ on \mathbb{B} for all $j \geq 1$, which yields that $\varphi|_{S_j \cap \mathbb{B}} = -\infty$ for all $j \geq 1$. Finally, set

$$u = \sup\{v \in \mathcal{PSH}(\Omega) : v \leq 0 \text{ and } v \leq \varphi \text{ on } \mathbb{B}\}.$$

By Lemma 4.5 in [29], Theorem 2.2 in [11] and Lemma 4.1 in [1], we get $u \in \mathcal{F}(\Omega)$, $(dd^c u)^n = c\delta_0$ and $\overline{\{u = -\infty\}} = \Omega$.

Proposition 2.2. *Assume that $\Omega \subseteq \mathbb{C}^n$ is a bounded hyperconvex domain. Let $u \in \mathcal{F}(\Omega)$ and $v \in \mathcal{PSH}(\Omega)$, $v \leq 0$, and $w \in \mathcal{E}(\Omega)$ be such that $(dd^c w)^n$ vanishes on pluripolar sets. If $(dd^c u)^n(\{u > -\infty\}) = 0$ and $u \geq v + w$ on a neighborhood D of $\{u = -\infty\}$ then $u \geq v$ on Ω .*

Proof. We have that

$$\max(u, v) + w = \max(u + w, v + w) \leq u,$$

and therefore by Lemma 4.1 in [1] we get that

$$(dd^c \max(u, v))^n \geq \chi_{\{u=-\infty\}}(dd^c u)^n = (dd^c u)^n.$$

Therefore, by Proposition 3.4 in [28] implies that $u = \max(u, v) \geq v$ on Ω . \square

3. A SUFFICIENT CONDITION ON μ

Lemma 3.1. *Assume that $\Omega_1 \subset C^{n_1}$ and $\Omega_2 \subset C^{n_2}$ are bounded hyperconvex domains. Let $u_1 \in \mathcal{E}(\Omega_1)$, $u_2 \in \mathcal{E}(\Omega_2)$ be such that*

$$(dd^c u_1)^n(\{u_1 > -\infty\}) = (dd^c u_2)^n(\{u_2 > -\infty\}) = 0.$$

Then

$$(dd^c \max(u_1, u_2))^{n_1+n_2} = (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}. \quad (3.1)$$

Proof. Set $u_1^j = \max(u_1, -j)$ and $u_2^j = \max(u_2, -j)$. From [7] (see also [3]), we have that

$$(dd^c \max(u_1^j, u_2^j))^{n_1+n_2} = (dd^c u_1^j)^{n_1} \wedge (dd^c u_2^j)^{n_2}.$$

By letting $j \rightarrow \infty$, we obtain that (3.1). \square

Lemma 3.2. *Let $\varphi \in \mathcal{E}(\Delta^k)$ be such that $(dd^c \varphi)^k(\{\varphi > -\infty\}) = 0$. Then*

$$(dd^c \max(\varphi(z_1, \dots, z_k), \log |z_{k+1}|, \dots, |z_n|))^n = (dd^c \varphi)^k \times \delta_0^{n-k}.$$

Proof. It follows from Lemma 3.1. \square

From Lemma 3.2 we have that

Theorem 3.3. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and S be a subvariety in Ω with dimension $k < n$. Assume that $\varphi \in \mathcal{E}(S)$ such that*

$$(dd^c \varphi)^k(\{\varphi > -\infty\}) = 0.$$

Then exists a function $u \in \mathcal{E}(\Omega)$ such that $(dd^c u)^n = (dd^c \varphi)^k$.

4. A NECESSARY CONDITION TO BELONG TO $\mathcal{E}(\Omega)$

In this section we start with introducing some notation. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $z' = (z_1, \dots, z_k)$ and $z'' = (z_{k+1}, \dots, z_n)$. Then we define

$$\begin{aligned}\|z\| &= \max(|z_1|, \dots, |z_n|), \\ \|z'\| &= \max(|z_1|, \dots, |z_k|), \\ \|z''\| &= \max(|z_{k+1}|, \dots, |z_n|),\end{aligned}$$

With these notation we make the following definition

Definition 4.1. Let $u \in \mathcal{PSH}(\Delta^n)$, $u \leq 0$. We define

$$\phi_u(z', r) = \frac{\max_{\|z''\|=r} u(z', z'')}{|\log r|},$$

and

$$\phi_u(z') = (-\nu_u(z', \cdot)(0))^*,$$

where $\nu_u(z', \cdot)(0)$ is the Lelong number of the function $u(z', \cdot)$ at 0.

From the construction in Definition 4.1 we get that $\phi_u(\cdot, r) \in \mathcal{PSH}(\Delta^k)$, $\phi_u(\cdot, r) \leq 0$ and that $\phi_u(z', r) \nearrow -\nu_u(z', \cdot)(0)$ as $r \searrow 0$. Thanks to [5], we have that $\phi_u \in \mathcal{PSH}(\Delta^k)$, $\phi_u \leq 0$, and that the set

$$\{z' \in \Delta^k : \phi_u(z') \neq -\nu_u(z', \cdot)(0)\}$$

is a pluripolar set in Δ^k . Furthermore, we get that

- if $u \geq v$, then $\phi_u \geq \phi_v$
- $\phi_{au+bv} = a\phi_u + b\phi_v$, for all $u, v \in \mathcal{PSH}(\Delta^n)$, $u, v \leq 0$, and $a, b \geq 0$
- $\phi_{\max(u,v)} = \max(\phi_u, \phi_v)$.

Theorem 4.2. Let $u \in \mathcal{PSH}(\Delta^n)$, $u \leq 0$. Then we have that ϕ_u is a constant function.

Proof. Take $z'_0 \in \Delta^k$. We will only need to prove that

$$\phi_u(z') \leq \phi_u(z'_0) \quad \text{for all } z' \in \Delta^k.$$

Fix $\epsilon > 0$. We can choose $r > 0$ small enough such that

$$\phi_u(z') \leq \phi_u(z'_0) + \epsilon \quad \text{for all } \|z' - z'_0\| < r.$$

This implies that

$$\nu_u(z', \cdot)(0) \geq -\phi_u(z'_0) - \epsilon \quad \text{for all } \|z' - z'_0\| < r.$$

Therefore, we have that

$$u(z', z'') \leq (-\phi_u(z'_0) - \epsilon) \log \|z''\| \quad \text{for } \|z' - z'_0\| < r, z'' \in \Delta^{n-k}.$$

Hence,

$$\{z' \in \Delta^k : \|z' - z'_0\| < r\} \times \{0\}^{n-k} \subset \{z \in \Delta^n : \nu_u(z) \geq -\phi_u(z'_0) - \epsilon\}.$$

On the other hand, from Siu's theorem (see e.g. [32, 13]) we have that

$$\{z \in \Delta^n : \nu_u(z) \geq -\phi_u(z'_0) - \epsilon\}$$

is an analytic set, which implies that

$$\{z \in \Delta^n : \nu_u(z) \geq -\phi_u(z'_0) - \epsilon\} = \Delta^k \times \{0\}^{n-k}.$$

Thus,

$$u(z', z'') \leq (-\phi_u(z'_0) - \epsilon) \log \|z''\| \quad \text{for all } z \in \Delta^n.$$

Hence, $\phi_u(z') \leq \phi_u(z'_0) + \epsilon$ for all $z' \in \Delta^k$. Let now $\epsilon \rightarrow 0^+$, and we finally get that

$$\phi_u(z') \leq \phi_u(z'_0) \quad \text{for all } z' \in \Delta^k.$$

□

Remark. If $k = n - 1$, then $(u - \phi_u \log \|z''\|) \in \mathcal{PSH}(\Delta^n)$.

Lemma 4.3. *Let u be a pluriharmonic function, and let $\{u_j\}$ be a sequence of plurisubharmonic functions that converges to u in dV_{2n} on Ω as $j \rightarrow \infty$. Then $\{u_j\}$ converges to u in capacity, as $j \rightarrow \infty$.*

Proof. Let $K \Subset L \Subset D \Subset \Omega$, and $\delta > 0$. We shall prove that

$$\text{Cap}_D(\{|u_j - u| > \delta\} \cap K) \rightarrow 0, \text{ as } j \rightarrow +\infty,$$

Choose $\phi \in \mathcal{E}_0(D)$ that satisfies $(dd^c \phi)^n = dV_{2n}$. Take $A > 0$ such that $A\phi \leq -1$ on L . Let $0 < \varepsilon < \frac{\delta}{2}$. Hartog's theorem yields that there exists a j_0 such that

$$u_j \leq u + \varepsilon \text{ for all } z \in D, \text{ and } j \geq j_0.$$

By Lemma 3.3 in [1], we have that

$$\begin{aligned} \text{Cap}_D(\{|u_j - u| > \delta\} \cap K) &= \text{Cap}_D(\{u - u_j > \delta\} \cap K) = \text{Cap}_D(\{u_j < u - \delta\} \cap K) \\ &= \sup \left\{ \int_{\{u_j < u - \delta\} \cap K} (dd^c \varphi)^n : \varphi \in \mathcal{PSH}(D), -1 \leq \varphi \leq 0 \right\} \\ &= \sup \left\{ \int_{\{u_j < u - \delta\} \cap K} (dd^c \varphi)^n : \varphi \in \mathcal{PSH}(D), h_{D,L}^* \leq \varphi \leq 0 \right\} \\ &\leq \frac{1}{\delta} \sup \left\{ \int_D (u - u_j + \varepsilon)(dd^c \varphi)^n : \varphi \in \mathcal{PSH}(D), h_{D,L}^* \leq \varphi \leq 0 \right\} \\ &\leq \frac{1}{\delta} \sup \left\{ \int_D (u - u_j + \varepsilon)(dd^c \varphi)^n : \varphi \in \mathcal{PSH}(D), A\phi \leq \varphi \leq 0 \right\} \\ &\leq \frac{1}{\delta} \int_D (u - u_j + \varepsilon)(dd^c A\phi)^n = \frac{A^n}{\delta} \int_D (u - u_j + \varepsilon)(dd^c \phi)^n \\ &= \frac{A^n}{\delta} \int_D (u - u_j + \varepsilon) dV_{2n} \leq \frac{A^n}{\delta} \left(\int_D |u - u_j| dV + \varepsilon V_{2n}(D) \right). \end{aligned}$$

Hence,

$$\limsup_{j \rightarrow +\infty} \text{Cap}_D(\{|u_j - u| > \delta\} \cap K) \leq \varepsilon \frac{A^n V_{2n}(D)}{\delta} \quad \text{for all } \varepsilon > 0.$$

Thus,

$$\text{Cap}_D(\{|u_j - u| > \delta\} \cap K) \rightarrow 0, \text{ as } j \rightarrow +\infty,$$

□

Theorem 4.4. *Let $u \in \mathcal{PSH}(\Delta^n)$, $u \leq 0$. Then we have that*

$$\frac{u(z', rz'')}{|\log r|} \rightarrow \phi_u$$

in capacity on $\Delta^k \times (\Delta_{\frac{1}{r}})^{n-k}$, as $r \rightarrow 0^+$. Here $\Delta_{\frac{1}{r}} \subseteq \mathbb{C}$ denotes the disc of radius $\frac{1}{r}$.

Proof. From

$$\phi_u(z', r) = \max_{\|z''\|=1} \frac{u(z', rz'')}{|\log r|} \nearrow \phi_u$$

as $r \rightarrow 0^+$, and Theorem 3.2.12 in [19] we get that

$$\frac{u(z', rz'')}{|\log r|} \rightarrow \phi_u \quad \text{in } dV_{2n} \text{ on } \Delta^k \times (\Delta_{\frac{1}{r}})^{n-k}$$

as $r \rightarrow 0^+$. We complete this proof by using Lemma 4.3 and obtain that

$$\frac{u(z', rz'')}{|\log r|} \rightarrow \phi_u$$

in capacity on $\Delta^k \times (\Delta_{\frac{1}{r}})^{n-k}$ as $r \rightarrow 0^+$. \square

Theorem 4.5. *Let $u \in \mathcal{E}(\Delta^n)$. Then ϕ_u is identically 0.*

Proof. Assume that $\phi_u < 0$. Hence

$$u(z', z'') \leq -\phi_u \log \|z''\| \quad \text{on } \Delta^n.$$

Hence, $\nu_u(z) \geq -\phi_u$ on $\Delta^k \times \{0\}^k$. This is not possible, since $u \in \mathcal{E}(\Delta^n)$. \square

Example 4.6 shows that the converse of Theorem 4.5 is in generally false.

Example 4.6. In this example we construct a function $u \in \mathcal{PSH}(\Omega)$, $u \leq 0$ such that

$$u(z) > -\infty \quad \text{for all } z \in \Omega$$

but $u \notin \mathcal{E}(\Omega)$. We can assume that $\Omega \subset \Delta^n$. Let u be defined on Δ^n as

$$u(z) = \sum_{j=1}^{\infty} \max \left(\frac{1}{2^j} \log \frac{|z_1 - \frac{1}{2^j}|}{|1 - \frac{z_1}{2^j}|}, \frac{2^j}{j} \log |z_2|, \log |z_3|, \dots, \log |z_n|, -2^j \right).$$

We start by proving that $u(z) > -\infty$ for all $z \in \Delta^n$. If $z_1 = 0$, then we have that

$$u(0) \geq \sum_{j=1}^{\infty} \frac{1}{2^j} \log \frac{1}{2^j} > -\infty.$$

If $z_1 \neq 0$ we choose j_0 be such that $|z_1| > \frac{1}{2^{j_0-1}}$. Hence

$$u(z) \geq \sum_{j=1}^{j_0} -2^j + \sum_{j=j_0+1}^{\infty} \frac{1}{2^j} \log \frac{|z_1 - \frac{1}{2^j}|}{|1 - \frac{z_1}{2^j}|} \geq \sum_{j=1}^{j_0} -2^j + \log \frac{|z_1|}{4} \sum_{j=j_0+1}^{\infty} \frac{1}{2^j} > -\infty.$$

Next, we shall show that $u \notin \mathcal{E}(W)$ for all neighbourhoods W of 0. Set

$$u_k = \sum_{j=1}^k \max \left(\frac{1}{2^j} \log \frac{|z_1 - \frac{1}{2^j}|}{|1 - \frac{z_1}{2^j}|}, \frac{2^j}{j} \log |z_2|, \log |z_3|, \dots, \log |z_n|, -2^j \right)$$

We have $u_k \in \mathcal{E}_0(\Delta^n)$ and $\varphi_k \searrow u$ as $k \rightarrow \infty$, and

$$\begin{aligned} (dd^c u_k)^n &\geq \sum_{j=1}^k \left(dd^c \max \left(\frac{1}{2^j} \log \frac{|z_1 - \frac{1}{2^j}|}{|1 - \frac{z_1}{2^j}|}, \frac{2^j}{j} \log |z_2|, \log |z_3|, \dots, \log |z_n|, -2^j \right) \right)^n \\ &= \sum_{j=1}^k \frac{1}{j} \sigma_{\{|z_1 - \frac{1}{2^j}| = e^{-4j}\}} \times \sigma_{\{|z_2| = e^{-j}\}} \times \sigma_{\{|z_3| = e^{-2j}\}} \cdots \times \sigma_{\{|z_n| = e^{-2j}\}}, \end{aligned}$$

where $\sigma_{\{|z_j| = r\}}$ is the normalized surface measure on $\{|z_j| = r\}$. Hence,

$$\int_W (dd^c u_k)^n \rightarrow \infty$$

as $k \rightarrow \infty$ for all neighbourhood W of 0.

Definition 4.7. For each $u \in \mathcal{PSH}(\Omega_1 \times \Omega_2)$ and $w_2 \in \Omega_2$ we define

$$\begin{aligned} E(u, t, w_2) &= \{z_1 \in \Omega_1 : u(z_1, z_2) \leq t \log \|z_2 - w_2\| + O(1), \text{ for every } z_2 \in \Omega_2\} \\ &= \{z_1 \in \Omega_1 : \nu_{u(z_1, \cdot)}(w_2) \geq t\}. \end{aligned}$$

Theorem 4.8. Let $u \in \mathcal{E}(\Omega_1 \times \Omega_2)$. Then

$$\bigcup_{t>0} E(u, t, w_2)$$

is a pluripolar set in Ω_1 for all $w_2 \in \Omega_2$.

Proof. Since this problem is purely local we can without loss of generality assume that $\Omega_1 = \Delta^k$, $\Omega_2 = \Delta^{n-k}$ and $w_2 = 0$. Theorem 4.5 yields that $\phi_u \equiv 0$. We have that

$$\bigcup_{t>0} E(u, t, 0) = \{z' \in \Delta^k : \phi_u(z') \neq -\nu_u(z', \cdot)(0)\},$$

and therefore it follows that $\bigcup_{t>0} E(u, t, 0)$ is a pluripolar set in Δ^k . \square

5. THE TORIC CASE

Theorem 5.1. Let $u \in \mathcal{E}(\Delta^n)$ be such that $u(z', z'') = u(z', |z_{k+1}|, \dots, |z_n|)$. Then there exists a Borel pluripolar set E in Δ^k such that

$$(dd^c u)^n((\Delta^k \setminus E) \times \{0\}^{n-k}) = 0.$$

Proof. Without loss generality we can assume that $u \in \mathcal{F}(\Delta^n)$. Theorem 6.3 in [10] yields that there exists a function $\varphi \in \mathcal{E}_0(\Delta^k)$, $0 \leq \varphi \in L^1((dd^c \varphi)^n)$, a non-negative Radon measure ν defined on Δ^k , and a Borel pluripolar set $E \subset \Delta^k$ such that

$$1_{\Delta^k \times \{0\}^{n-k}}(dd^c u)^n = f(dd^c \varphi)^k + \nu,$$

and $\nu(\Delta^k \setminus E) = 0$. We shall prove that

$$f(dd^c \varphi)^k = 0.$$

Fix $t \in (0, 1)$. Thanks to Lemma 4.3 in [1], we can find a function $v \in \mathcal{F}(\Delta^n)$ such that $v \geq u$, $(dd^c v)^n = 1_{\Delta^n} f(dd^c \varphi)^k$ and

$$v(z', z'') = v(z', |z_{k+1}|, \dots, |z_n|).$$

Next choose a sequence $\{r_j\}$ with $r_j \searrow 0$. By the quasicontinuity of $\phi_v(\cdot, r_j)$ (see e.g. [5]), we can find a decreasing sequence of open sets $\{G_m\}_{m \geq 1}$ in Δ_t^k such that

$$\text{Cap}_{\Delta^k}(G_m) < \frac{1}{m} \quad \text{and} \quad \phi_v(\cdot, r_j)|_{\overline{\Delta}_t^k \setminus G_m} \text{ are continuous.}$$

Furthermore, it can be chosen such that each element is continuous on $\overline{\Delta}_t^k \setminus G_m$ for all $j, m \geq 1$, and $\nu_{v(z', \cdot)}(0) = 0$ on $\overline{\Delta}_t^k \setminus G_m$ for all $m \geq 1$. By Dini's theorem we have $\phi_v(z', r_j)$ converges uniformly 0 on $z' \in \overline{\Delta}_t^k \setminus G_m$, as $j \rightarrow \infty$, for all $m \geq 1$. Hence, for each m we can choose j_m such that

$$\epsilon_m = - \min_{z' \in \overline{\Delta}_t^k \setminus G_m} \phi_v(z', r_{j_m}) \searrow 0, \quad \text{as } m \rightarrow \infty.$$

Since $v(z', z'') = v(z', |z_{k+1}|, \dots, |z_n|)$, we have that

$$v(z', z'') \geq \epsilon_{j_m} (\log |z_{k+1}| + \dots + \log |z_n|),$$

for all $(z', z'') \in (\overline{\Delta}_t^k \setminus G_m) \times \overline{\Delta}_{r_{j_m}}^{n-k}$. Set

$$w_m = \max(v, \epsilon_{j_m} (\log |z_{k+1}| + \dots + \log |z_n|)),$$

and choose $v_l \in \mathcal{E}_0 \cap C(\Delta^n)$ such that $v_l \searrow v$. Set

$$h_{G_m, \Delta^k} = \sup \{ \varphi \in \mathcal{PSH}(\Delta^k) : \varphi \leq -1 \text{ on } G_m \}$$

We have that

$$\begin{aligned} & \int_{\overline{\Delta}_t^k \setminus G_m \times \overline{\Delta}_{r_{j_m}}^{n-k}} (dd^c w_m)^n \\ & \geq \overline{\lim}_{l \rightarrow \infty} \int_{\overline{\Delta}_t^k \setminus G_m \times \overline{\Delta}_{r_{j_m}}^{n-k}} \left(dd^c \max(v_l, \epsilon_{j_m} (\log |z_{k+1}| + \dots + \log |z_n|) - \frac{1}{l}) \right)^n. \end{aligned}$$

Since,

$$\overline{\Delta}_t^k \setminus G_m \times \overline{\Delta}_{r_{j_m}}^{n-k} \subset \left\{ v_l > \epsilon_{j_m} (\log |z_{k+1}| + \dots + \log |z_n|) - \frac{1}{l} \right\}$$

and $h(dd^c v_l)^n \rightarrow h(dd^c v)^n$ weakly as $l \rightarrow \infty$ for all $h \in \mathcal{PSH}(\Delta^n) \cap L^\infty(\Delta^n)$, we have that

$$\begin{aligned} & \int_{\overline{\Delta}_t^k \setminus G_m \times \overline{\Delta}_{r_{j_m}}^{n-k}} (dd^c w_m)^n \geq \overline{\lim}_{l \rightarrow \infty} \int_{\overline{\Delta}_t^k \setminus G_m \times \overline{\Delta}_{r_{j_m}}^{n-k}} (dd^c v_l)^n \\ & \geq \underline{\lim}_{l \rightarrow \infty} \int_{\Delta_t^k \times \Delta_{r_{j_m}}^{n-k}} (1 + h_{G_m, \Delta^k})(dd^c v_l)^n \geq \int_{\Delta_t^k \times \Delta_{r_{j_m}}^{n-k}} (1 + h_{G_m, \Delta^k})(dd^c v)^n \\ & = \int_{\Delta_t^k} (1 + h_{G_m, \Delta^k}) f(dd^c \varphi)^k. \end{aligned}$$

From the fact that

$$\int_{\Delta^k} (dd^c h_{G_m, \Delta^k})^n = \text{Cap}_{\Delta^k}(G_m) \searrow 0 \quad \text{as } m \rightarrow \infty$$

we get $h_{G_m, \Delta^k} \nearrow 0$ a.e on Δ^k , as $m \rightarrow \infty$. This yields that

$$\underline{\lim}_{m \rightarrow \infty} \int_{\overline{\Delta}_t^k \setminus G_m \times \overline{\Delta}_{r_{j_m}}^{n-k}} (dd^c w_m)^n \geq \int_{\Delta_t^k} f(dd^c \varphi)^k.$$

On the other hand, since $v \leq w_m \nearrow 0$ as $m \rightarrow \infty$, we get that

$$\overline{\lim}_{m \rightarrow \infty} \int_{\Delta_t^k \times \overline{\Delta}_{r_{j_m}}^{n-k}} (dd^c w_m)^n \leq \overline{\lim}_{m \rightarrow \infty} \int_{\Delta_t^k \times \overline{\Delta}_{r_{j_1}}^{n-k}} (dd^c w_m)^n \leq 0$$

Thus,

$$\int_{\Delta_t^k} f(dd^c \varphi)^k = 0$$

To complete this proof let $t \rightarrow 1^-$. \square

By combining Theorem 5.1 with Theorem 3.3 we get the following corollary

Corollary 5.2. *Let μ be a non-negative Radon measure defined on Δ^k which vanish on every pluripolar sets in Δ^k . Then there is exists no function $u \in \mathcal{E}(\Delta^n)$ such that*

$$u(z', z'') = u(z', |z_{k+1}|, \dots, |z_n|) \quad \text{and} \quad (dd^c u)^n = \mu.$$

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